HJB Equations for the Optimal Control of Differential Equations with Delays and State Constraints, I: Regularity of Viscosity Solutions. *

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July 1, 2010

Abstract

We study a class of optimal control problems with state constraints, where the state equation is a differential equation with delays. This class includes some problems arising in economics, in particular the so-called models with time to build, see [1, 2, 26]. We embed the problem in a suitable Hilbert space H and consider the associated Hamilton-Jacobi-Bellman (HJB) equation. This kind of infinite-dimensional HJB equation has not been previously studied and is difficult due to the presence of state constraints and the lack of smoothing properties of the state equation. Our main result on the regularity of solutions to such a HJB equation seems to be entirely new. More precisely, we prove that the value function is continuous in a sufficiently big open set of H, that it solves in the viscosity sense the associated HJB equation and it has continuous classical derivative in the direction of the "present". This regularity result is the starting point to define a feedback map in classical sense, which gives rise to a candidate optimal feedback strategy.

Keywords: Hamilton-Jacobi-Bellman equation, optimal control, delay equations, viscosity solutions, regularity.

A.M.S. Subject Classification: 34K35,49L25, 49K25.

*This work was partially supported by an Australian Research Council Discovery Project

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1 Introduction

The main purpose of this paper is to prove a C^1 regularity result for a class of first order infinite dimensional HJB equations associated to the optimal control of deterministic delay equations arising in economic models.

The C^1 regularity of solutions to the HJB equations arising in deterministic optimal control theory is a crucial step in solving the control problems. Indeed, in order to obtain an optimal strategy in feedback form one needs the existence of an appropriately defined gradient of the solution. It is possible to prove verification theorems and representation of optimal feedbacks in the framework of viscosity solutions, even if the gradient is not defined in classical sense (see e.g. [8, 31]), but this is usually not satisfactory in applied problems since the closed loop equation becomes very hard to treat in such cases.

The C^1 regularity of solutions to HJB equations is particularly important in infinite dimension since in this case verification theorems in the framework of viscosity solutions are rather weak and in any case not applicable to problems with state constraints (see e.g [18, 27]). To the best of our knowledge, C^1 regularity for first order HJB equation was proved by method of convex regularization introduced by Barbu and Da Prato [3] and then developed by various authors (see e.g. [4, 5, 6, 7, 15, 16, 19, 22, 23]). All these results do not hold in the case of state constraints and, even without state constraints, do not cover problems where the state equation is a nonlinear differential equation with delays. In the papers [10, 12, 20] a class of problems with state constraints is treated using the method of convex regularization but the C^1 regularity is not proved.

In this paper we deal with a class of optimal control problems where, given a control $c(\cdot) \geq 0$

the state $x(\cdot)$ satisfies the following delay equation

$$\begin{cases} x'(t) = rx(t) + f_0\left(x(t), \int_{-T}^{0} a(\xi)x(t+\xi)d\xi\right) - c(t), \\ x(0) = \eta_0,\ x(s) = \eta_1(s),\ s \in [-T, 0), \end{cases}$$

with state constraint $x(\cdot) > 0$ or $x(\cdot) \geq 0$. Given two functions $U_1 : \mathbb{R}^+ \to \mathbb{R}^+$, $U_2 : \mathbb{R}^+ \to [-\infty, +\infty)$, both increasing and concave, the objective is to maximize the functional

$$J(\eta; c(\cdot)) := \int_0^{+\infty} e^{-\rho t}\big(U_1(c(t)) + U_2(x(t))\big)\, dt, \quad \rho > 0,$$

over the set of the admissible controls $c(\cdot)$. We may think of the functions U_1, U_2 as suitable utility functions, see Section 2 for more details. Problems of this type arise in various economic models. In particular, in [1, 2, 26] the authors study optimal growth in presence of time-to-build (i.e. delay in the production due to the need of time to build new products) that cannot be studied using the existing theory except for very special cases, see [1, 2, 26] again.

Using a standard approach (see e.g. [9]) we reformulate our problem as a control problem in the Hilbert space $H = \mathbb{R} \times L^2([-T, 0]; \mathbb{R})$ where, intuitively speaking, $\mathbb{R}$ describes the "present" and $L^2([-T, 0]; \mathbb{R})$ describes the "past" of the system. The associated Hamilton-Jacobi-Bellman equation in H has not been previously studied and is difficult due to the presence of state constraints and the lack of smoothing properties of the state equation.

The cases with state constraint $x(\cdot) > 0$ and the state constraint $x(\cdot) \geq 0$ are different; we consider mainly the first one and then devote Section 5 to explain which results remain true for the second one. Concerning the first case we prove that the value function is continuous in a sufficiently big open set of H (Proposition 3.8), that it solves in the viscosity sense the associated HJB equation (Theorem 4.4) and it has continuous classical derivative in the direction of the "present" (Theorem 4.6). This regularity result allows us to define the formal optimal feedback strategy in classical sense, since the objective functional depends on the "present" only. The method we use to prove regularity is entirely different from the one of convex regularization mentioned above. Indeed, it is based on a finite dimensional result of Cannarsa and Soner [13] (see also [8], pag. 80) that exploits the concavity of the data and the strict convexity of the Hamiltonian to prove the continuous differentiability of the viscosity solution of the HJB equation. Generalizing such a result to the infinite dimensional case is not trivial as the definition of viscosity solution in this case strongly depends on the unbounded differential operator A which appears in the state equation. In particular, we need to establish specific properties of superdifferential that are given in Subsection 3.3.

We believe that such a method could be also used to analyze other problems with concavity of the data and strict convexity of the Hamiltonian.

The plan of the paper is as follows. Section 2 is devoted to set up the problem in DDE form giving main assumptions and some preliminary results (in Subsection 2.1) that are proved directly without using the infinite dimensional setting. In Section 3 we rewrite the problem in the infinite dimensional setting and prove the existence and uniqueness of solutions of the state equation (Subsection 3.1), continuity of the value function (Subsection 3.2) and some useful properties of superdifferentials (Subsection 3.3). In Section 4 we apply the dynamic programming in infinite dimensional context to obtain our main results: we prove that the value function is a viscosity solution of the HJB equation (Subsection 4.1) and then we prove a regularity result for viscosity solutions of HJB (Subsection 4.2). In Section 5 we explain which results hold in the case when the state constraint $x(\cdot) > 0$ is substituted by $x(\cdot) \geq 0$.

2 Formulation of the control problem and preliminary results

In this section we define the control delay problem and provide some financial motivations for it. We will use the notations

$$L^p_T := L^p([-T,0];\mathbb{R}),\ p \geq 1, \quad \text{and} \quad W^{1,2}_T := W^{1,2}([-T,0];\mathbb{R}).$$

We will denote by H the Hilbert space

$$H := \mathbb{R} \times L^2_T,$$

endowed with the inner product $\langle \cdot,\cdot \rangle$ defined by

$$\langle \cdot,\cdot \rangle = \langle \cdot,\cdot \rangle_{\mathbb{R}} + \langle \cdot,\cdot \rangle_{L^2_T},$$

and the norm $\| \cdot \|$ defined by

$$\| \cdot \|^2 = | \cdot |^2_{\mathbb{R}} + \| \cdot \|^2_{L^2_T}.$$

We will denote by $\eta = (\eta_0, \eta_1(\cdot))$ a generic element of this space. For convenience we also set

$$H_+ := (0,+\infty) \times L^2_T, \qquad H_{++} := (0,+\infty) \times \{\eta_1(\cdot) \in L^2_T \mid \eta_1(\cdot) \geq 0 \text{ a.e.}\},$$

$$\bar{H}_+ := [0,+\infty) \times L^2_T, \qquad \bar{H}_{++} := [0,+\infty) \times \{\eta_1(\cdot) \in L^2_T \mid \eta_1(\cdot) \geq 0 \text{ a.e.}\}.$$

Remark 2.1. Economic motivations we are mainly interested in (see [1, 2, 26] and Remark 2.7 below) require to study the optimal control problem with the initial condition in H_{++} in the case of state constraint $x(\cdot) > 0$ or in $\bar{H}_+$ in the case of state constraint $x(\cdot) \geq 0$. However the sets H_{++} and $\bar{H}_{++}$ are not convenient to work with, since their interior with respect to the $\| \cdot \|$-norm is empty. This is why we take initial states belonging to H_+ or $\bar{H}_+$ (respectively in the case of state constraint $x(\cdot) > 0$ or $x(\cdot) \geq 0$). ■

For $\eta \in H_+$ (respectively, $\eta \in \bar{H}_+$), we consider the following controlled differential delay equation:

$$\begin{cases} x'(t) = rx(t) + f_0\left(x(t), \int_{-T}^0 a(\xi)x(t+\xi)d\xi\right) - c(t), \\ x(0) = \eta_0,\ x(s) = \eta_1(s),\ s \in [-T,0), \end{cases} \tag{1}$$

with the state constraint $x(\cdot) > 0$ (respectively, $x(\cdot) \geq 0$) and control constraint $c(\cdot) \geq 0$.

The following will be standing assumptions on the functions a, f_0. They will hold throughout the whole paper and will not be repeated.

Hypothesis 2.2.

(i) $a(\cdot) \in W^{1,2}_T$ is such that $a(\cdot) \geq 0$ and $a(-T) = 0$;

(ii) $f_0 : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ is jointly concave, nondecreasing with respect to the second variable, Lipschitz continuous with Lipschitz constant C_{f_0}, and

$$f_0(0,0) \geq 0. \tag{2}$$

■

Remark 2.3. In papers [1, 2, 26] the point delay is used. We are not able to treat this case for technical reasons that are explained in Remark 4.9 below. However, we have the freedom of choosing the function a from a wide class and this allows us to take into account various economic phenomena. Moreover, we can approximate a point delay with a suitable sequence of functions $\{a_n\}$ getting convergence of the value functions and constructing ε-optimal strategies. This approximation procedure is an object of a forthcoming paper. ■

From now on we will assume that f_0 is extended to a Lipschitz continuous map on $\mathbb{R}^2$ setting

$$f_0(x,y) := f_0(0,y), \quad \text{for } x<0.$$

For technical reasons, which will become clear in Subsection 3.2, we work with the case $r>0$, noting that the case $r \le 0$ can be treated as well by shifting the linear part of the state equation. Indeed, in this case we can rewrite the state equation taking for example $\tilde{r}=1$ as a new coefficient for the linear part and shifting the nonlinear term to obtain $\tilde{f}_0(x,y) = f_0(x,y) - (1-r)x$.

We say that a function $x : [-T,\infty) \longrightarrow \mathbb{R}^+$ is a solution to equation (1) if $x(t) = \eta_1(t)$ for $t \in [-T,0)$,

$$\int_0^t |x(s)|ds < \infty, \quad t \ge 0,$$

and

$$x(t) = \eta_0 + \int_0^t rx(s)ds + \int_0^t f_0\left(x(s), \int_{-T}^0 a(\xi)x(s+\xi)d\xi\right)ds - \int_0^t c(s)ds, \quad t \ge 0. \tag{3}$$

Theorem 2.4. *For any $\eta \in H$, $c(\cdot) \in L^1_{loc}([0,+\infty);\mathbb{R}^+)$, equation* (1) *admits a unique solution that is absolutely continuous on* $[0,+\infty)$.

Proof. Let $K = \sup_{\xi\in[-T,0]} a(\xi)$. For any $t \ge 0$, $z^1, z^2 \in C([-T,t];\mathbb{R})$, we have

$$\begin{aligned}
&\int_0^t \left[r|z_1(s)-z_2(s)| + \left|f_0\left(z_1(s), \int_{-T}^0 a(\xi)z_1(s+\xi)\right) - f_0\left(z_2(s), \int_{-T}^0 a(\xi)z_2(s+\xi)\right)\right|\right] ds \\
&\quad \le \int_0^t \left[r|z_1(s)-z_2(s)| + C_{f_0}\left[|z_1(s)-z_2(s)| + K\int_{-T}^0 |z_1(s+\xi)-z_2(s+\xi)|d\xi\right]\right] ds \\
&\quad \le \int_0^t \left[(r+C_{f_0})|z_1(s)-z_2(s)| + C_{f_0}K\int_{-T}^t |z_1(\xi)-z_2(\xi)|d\xi\right] ds \\
&\quad \le (r+C_{f_0})\int_0^t |z_1(s)-z_2(s)|ds + tC_{f_0}K\int_{-T}^t |z_1(\xi)-z_2(\xi)|d\xi \\
&\quad \le \left[(r+C_{f_0}) + tC_{f_0}K\right]\int_{-T}^t |z_1(\xi)-z_2(\xi)|d\xi \\
&\quad \le \left[(r+C_{f_0}) + tC_{f_0}K\right](t+T)^{1/2}\left(\int_{-T}^t |z_1(\xi)-z_2(\xi)|^2 d\xi\right)^{1/2}.
\end{aligned}$$

Thus the claim follows by Theorem 3.2, p. 246, of [9]. □

We denote by $x(\cdot;\eta,c(\cdot))$ the unique solution of (1) with initial value $\eta \in H_+$ and control $c(\cdot)$. We emphasize that this is a solution to integral equation (3); it satisfies differential equation (1) only for almost every $t \in [0, +\infty)$.

For an initial condition $\eta \in H_+$ we define a class of the admissible controls for the problem with state constraint $x(\cdot) > 0$ as

$$\mathcal{C}(\eta) := \{c(\cdot) \in L^1_{loc}([0, +\infty); \mathbb{R}^+) \mid x(\cdot; \eta, c(\cdot)) > 0\}. \tag{4}$$

In analogous way, for an initial condition $\eta \in \bar{H}_+$ we define a class of the admissible controls for the problem with state constraint $x(\cdot) \geq 0$ as

$$\bar{\mathcal{C}}(\eta) := \{c(\cdot) \in L^1_{loc}([0, +\infty); \mathbb{R}^+) \mid x(\cdot; \eta, c(\cdot)) \geq 0\}. \tag{5}$$

In both cases, setting $x(\cdot) := x(\cdot; \eta, c(\cdot))$, the problem consists in maximizing the functional

$$J(\eta; c(\cdot)) := \int_0^{+\infty} e^{-\rho t} \big(U_1(c(t)) + U_2(x(t)) \big)\, dt, \tag{6}$$

over the set of the corresponding admissible strategies.

From now on we consider the first problem, that is the one with state constraint $x(\cdot) > 0$. We refer to Section 5 for comments on the case with state constraint $x(\cdot) \geq 0$.

The following will be standing assumptions on the utility functions U_1, U_2 and on the disconuting rate ρ. They will hold throughout the whole paper and will not be repeated.

Hypothesis 2.5.

(i) $U_1 \in C([0, +\infty); \mathbb{R}) \cap C^2((0, +\infty); \mathbb{R})$ and

$$U_1' > 0, \; U_1'' < 0; \;\; U_1'(0^+) = +\infty; \tag{7}$$

$$\exists\, \beta_1 \in [0, 1), C_1 > 0 \text{ such that } U_1(c) \leq C_1(1 + c^{\beta_1}). \tag{8}$$

Without loss of generality we will assume $U_1(0) = 0$. We note that (7) and (8) imply $\lim_{c \to +\infty} U_1'(c) = 0$.

(ii) $U_2 : [0, +\infty) \to [-\infty, +\infty)$, $U_2 \in C((0, +\infty); \mathbb{R})$ is increasing and concave. Moreover

$$\int_0^{+\infty} e^{-\rho t} U_2 \left(e^{-C_{f_0} t} \right) dt > -\infty. \tag{9}$$

and

$$\exists\, \beta_2 \in [0, 1), C_2 > 0 \text{ such that } U_2(x) \leq C_2(1 + x^{\beta_2}). \tag{10}$$

(iii) The discounting rate ρ is such that

$$\rho > (\beta_1 \vee \beta_2) \left(r + C_{f_0} \left(1 + T \cdot \sup_{\xi \in [-T, 0]} a(\xi) \right) \right), \tag{11}$$

where C_{f_0} is the Lipschitz constant of f_0 and $\beta_1, \beta_2 \in [0, 1)$ are the constants in (8) and (10), respectively. ■

Remark 2.6. We give some comments on Hypothesis 2.5.

1. Utility functions such as $\frac{c^\gamma}{\gamma}$, $\gamma \in (0,1)$, are possible choices for U_1. Utility functions such as $\frac{x^\gamma}{\gamma}$, $\gamma \in (-\rho C_{f_0}^{-1}, 0) \cup (0,1)$, or $\log(x)$ are possible choices for U_2.
2. Through the whole paper the case $U_2 \equiv 0$ is allowed. Therefore, the case of an objective functional depending only on consumption (as in [1, 2, 26]) is allowed.
3. Note that (9) is equivalent to
$$\int_0^{+\infty} e^{-\rho t} U_2\left(\xi e^{-C_{f_0} t}\right) dt > -\infty, \quad \forall \xi > 0.$$
4. If $r < 0$, then in (9) we have to replace C_{f_0} with $|r| + C_{f_0}$.
5. If we assume that
$$\exists \delta > 0 \text{ such that } rx + f_0(x,0) \geq 0, \forall x \in (0,\delta], \tag{12}$$
then assumption (9) can be removed. Due to (2), we see that (12) holds for example if $x \mapsto rx + f_0(x,0)$ is nondecreasing. Therefore in particular it holds if $r \geq 0$ and f_0 depends only on the second variable (see the analogy of this case with [1, 2, 26]). ■

Remark 2.7. We believe that our technique could be adapted to cover optimal advertising and optimal investment/consumption models with nonlinear memory effects. We refer to [21] for a survey on optimal advertising models (where the introduction of memory effects is advocated) and [24] for a treatment of such problems in stochastic environment. ■

2.1 Preliminary results

In this subsection we investigate some qualitative properties of state equation (1), of the set of admissible strategies (4), of the objective functional (6) and of the value function of our optimal control problem.

2.1.1 State equation

We prove here a useful comparison result for state equation (1).

Lemma 2.8 (Comparison). *Let $\eta \in H$ and let $c(\cdot) \in L^1_{loc}([0,+\infty);\mathbb{R}^+)$. Let $x(t)$, $t \geq 0$, be an absolutely continuous function satisfying almost everywhere the differential inequality*

$$\begin{cases} x'(t) \leq rx(t) + f_0\left(x(t), \int_{-T}^0 a(\xi)x(t+\xi)d\xi\right) - c(t), \\ x(0) \leq \eta_0, \ x(s) \leq \eta_1(s), \ \text{for a.e. } s \in [-T,0). \end{cases}$$

Then $x(\cdot) \leq x(\cdot;\eta,c(\cdot))$.

Proof. Set $\bar{a} := \sup_{\xi\in[-T,0]} |a(\xi)|$, $y(\cdot) := x(\cdot;\eta,c(\cdot))$ and $h(\cdot) := [x(\cdot) - y(\cdot)]^+$. We show that $h(\cdot) = 0$. Let $\varepsilon > 0$ be such that $\varepsilon C_{f_0} \bar{a} T e^{\varepsilon(r+C_{f_0})} \leq 1/2$ and let $M := \max_{t\in[0,\varepsilon]} h(t)$. By monotonicity with respect to the second variable of f_0 (Hypothesis 2.2-(ii)) we get

$$f_0\left(x(t), \int_{-T}^0 a(\xi)x(t+\xi)d\xi\right) \leq f_0\left(x(t), \int_{-T}^0 a(\xi)y(t+\xi)d\xi + \bar{a}TM\right), \quad \text{for } t \in [0,\varepsilon]. \tag{13}$$

Define, for $n \in \mathbb{N}$,

$$\varphi_n(x) := \begin{cases} 0, & \text{for } x \leq 0, \\ nx^2, & \text{for } x \in (0, 1/2n], \\ x - 1/4n, & \text{for } x > 1/2n. \end{cases}$$

and observe that the sequence $(\varphi_n)_{n\in\mathbb{N}} \subset C^1(\mathbb{R};\mathbb{R})$ is such that

$$\begin{cases} \varphi_n(x) = \varphi_n'(x) = 0, & \text{for every } x \in (-\infty, 0], \ n \in \mathbb{N}, \\ 0 \leq \varphi_n'(x) \leq 1, & \text{for every } x \in \mathbb{R}, \ n \in \mathbb{N}, \\ \varphi_n(x) \to x^+, & \text{uniformly on } x \in \mathbb{R}, \\ \varphi_n'(x) \to 1, & \text{for } x \in (0, +\infty). \end{cases}$$

Now, taking into account (13), we have, for $t \in [0, \varepsilon]$,

$$\begin{aligned} \varphi_n(x(t) - y(t)) &= \varphi_n(x(0) - \eta_0) + \int_0^t \varphi_n'(x(s) - y(s))[x'(s) - y'(s)]ds \\ &\leq \int_0^t \varphi_n'(x(s) - y(s)) \Bigg[r(x(s) - y(s)) \\ &\quad + f_0\left(x(s), \int_{-T}^0 a(\xi)x(s+\xi)d\xi\right) - f_0\left(y(s), \int_{-T}^0 a(\xi)y(s+\xi)d\xi\right)\Bigg] ds \\ &\leq \int_0^t \varphi_n'(x(s) - y(s)) \Bigg[r(x(s) - y(s)) \\ &\quad + f_0\left(x(s), \int_{-T}^0 a(\xi)y(s+\xi)d\xi + \bar{a}TM\right) - f_0\left(y(s), \int_{-T}^0 a(\xi)y(s+\xi)d\xi\right)\Bigg] ds \\ &\leq \int_0^t \varphi_n'(x(s) - y(s)) \Big[(r + C_{f_0})|x(s) - y(s)| + C\bar{a}TM \Big] ds. \end{aligned}$$

Letting $n \to \infty$ we get

$$h(t) \leq \int_0^t (r + C_{f_0})\, h(s) ds + C_{f_0}\bar{a}TMt \leq \int_0^t (r + C_{f_0})\, h(s) ds + C_{f_0}\bar{a}TM\varepsilon.$$

Therefore by the Gronwall Lemma we get

$$h(t) \leq \varepsilon C_{f_0}\bar{a}TMe^{\varepsilon(r+C_{f_0})}, \quad \text{for } t \in [0, \varepsilon],$$

so, using the definition of ε,

$$h(t) \leq \frac{M}{2}, \quad \text{for } t \in [0, \varepsilon].$$

This shows that $M = 0$, i.e. that $h = 0$ on $[0, \varepsilon]$. Iterating the argument, since ε is fixed, we get $h \equiv 0$ on $[0, +\infty)$, i.e. the claim. □

2.1.2 Admissible strategies

Here we prove two useful properties of the set of admissible strategies (4).

Proposition 2.9.

1. *For every $\eta \in H_+$, we have $\mathcal{C}(\eta) \neq \emptyset$ if and only if $0 \in \mathcal{C}(\eta)$.*

2. *For every $\eta \in H_{++}$ we have $x(t;\eta,0) \geq \eta_0 e^{-C_{f_0} t}$ for all $t \geq 0$, where C_{f_0} is the Lipschitz constant of f_0. In particular, we have $\mathcal{C}(\eta) \neq \emptyset$ for every $\eta \in H_{++}$.*

Proof. 1. This claim follows immediately from Lemma 2.8.

2. Let $\eta \in H_{++}$ and set $x(\cdot) := x(\cdot;\eta,0)$. Since $x(0) = \eta_0 > 0$, we see that $x(t)$ remains strictly positive for t in a right neighborhood of 0. Let

$$t_1 = \inf\{t > 0 \mid x(t) = 0\}.$$

Clearly $t_1 > 0$. By monotonicity of f_0 with respect to the second variable (Hypothesis 2.2-(ii)), we have for $t \in [0, t_1]$:

$$x'(t) = rx(t) + f_0\left(x(t), \int_{-T}^{0} a(\xi)x(t+\xi)d\xi\right) \geq rx(t) + f_0(x(t), 0).$$

Since $f_0(0,0) \geq 0$ and $f_0(\cdot, 0)$ is Lipschitz continuous (with Lipschitz constant C_{f_0}), we get

$$x'(t) \geq -C_{f_0}\, x(t), \quad t \in [0, t_1].$$

This fact implies $t_1 = +\infty$ and $x(t) \geq \eta_0 e^{-C_{f_0} t}$ for any $t \geq 0$. □

2.1.3 Objective functional

In the next proposition we give upper bounds for the state $x(\cdot)$ and for the functional defined in (6).

Proposition 2.10.

1. *For every $\eta \in H_+$, there exist constants $K_0, K_\eta > 0$ such that*

$$x(t;\eta,0) \leq K_\eta e^{K_0 t}. \tag{14}$$

2. *For every $\eta \in H_+$, there exists $C_\eta > 0$ such that*

$$\int_0^{+\infty} e^{-\rho t}\big(U_1(c(t)) + U_2^+(x(t))\big)dt \leq C_\eta < +\infty, \quad \forall c(\cdot) \in \mathcal{C}(\eta). \tag{15}$$

In particular, the functional (6) is well defined[1] *for every $\eta \in H_+$, $c(\cdot) \in \mathcal{C}(\eta)$.*

Proof. 1. Let

$$\bar{a} := \sup_{\xi \in [-T,0]} a(\xi), \qquad p := f_0(0,0) \geq 0.$$

Since f_0 is Lipschitz continuous with Lipschitz constant C_{f_0}, we have

$$rx + f_0(x,y) \leq rx + C_{f_0}(x + |y|) + p := g(x,y), \quad \forall x \in \mathbb{R}^+,\ \forall y \in \mathbb{R}. \tag{16}$$

[1] Even if it may take value $-\infty$.

Let $z(\cdot)$ be the solution to the problem

$$\begin{cases} z'(t) = g\left(z(t), \int_{-T}^{0} a(\xi) z(t+\xi) d\xi\right), \\ z(0) = \eta_0, \ z(s) = \eta_1(s), \ \text{for a.e. } s \in [-T, 0). \end{cases}$$

Of course $x(t;\eta,0) \le z(\cdot)$ by Lemma 2.8. Since g is positive, we see that $z(\cdot)$ is positive and nondecreasing. So

$$z'(t) = g\left(z(t), \int_{-T}^{0} a(\xi) z(t+\xi) d\xi\right) \le r z(t) + C_{f_0}\left(z(t) + \bar{a}\|\eta_1\|_{L^1_{-T}} + \bar{a} T z(t)\right) + p.$$

Therefore, the Gronwall Lemma yields

$$x(t;\eta,0) \le z(t) \le \left(\eta_0 + \frac{C_{f_0}\bar{a}\|\eta_1\|_{L^1_{-T}} + p}{r + C_{f_0}(1+\bar{a}T)}\right) e^{\left(r + C_{f_0}(1+\bar{a}T)\right)t},$$

and the claim follows with

$$K_\eta = \eta_0 + \frac{C_{f_0}\bar{a}\|\eta_1\|_{L^1_{-T}} + p}{r + C_{f_0}(1+\bar{a}T)}, \quad K_0 = r + C_{f_0}(1+\bar{a}T).$$

2. Let $\eta \in H_+$, $c(\cdot) \in \mathcal{C}(\eta)$ and set $x(\cdot) := x(\cdot;\eta, c(\cdot))$. By Lemma 2.8, estimate (14) and the admissibility of $c(\cdot)$, we have

$$0 \le x(t) \le x(t;\eta,0) \le K_\eta e^{K_0 t}, \qquad \forall t \ge 0. \tag{17}$$

Let us estimate the part with U_2^+ in (15). From (10) we get

$$\int_0^{+\infty} e^{-\rho t} U_2^+(x(t)) dt \le \frac{C_2}{\rho} + C_2 \int_0^{+\infty} e^{-\rho t} x(t)^{\beta_2} dt.$$

Then, using (17),

$$\int_0^{+\infty} e^{-\rho t} U_2^+(x(t)) dt \le K'_\eta \left(1 + \int_0^{+\infty} e^{-\rho t} e^{K_0 \beta_2 t} dt\right), \tag{18}$$

where $K'_\eta = \max\left\{\frac{C_2}{\rho}, C_2 K_\eta^{\beta_2}\right\}$.

Now let us estimate the part with U_1 in (15). By (1) and (16) we have

$$x'(t) \le g\left(x(t), \int_{-T}^{0} a(\xi) x(t+\xi) d\xi\right) - c(t).$$

Using (17) in the right hand side of the above estimate, we get

$$x'(t) \le r K_\eta e^{K_0 t} + C_{f_0}\left(K_\eta e^{K_0 t} + \bar{a}\|\eta_1\|_{L^1_{-T}} + \bar{a} T K_\eta e^{K_0 t}\right) + p - c(t), \quad \text{for a.e. } t \ge 0.$$

Integrating the above inequality and taking into account that $c(\cdot) \in \mathcal{C}(\eta)$ yields $x(t) > 0$ for every $t \ge 0$, we get

$$\int_0^t c(s) ds \le \eta_0 + \int_0^t \left(r K_\eta e^{K_0 s} + C_{f_0}\left(K_\eta e^{K_0 s} + \bar{a}\|\eta_1\|_{L^1_{-T}} + \bar{a} T K_\eta e^{K_0 s}\right) + p\right) ds, \quad \forall t \ge 0.$$

So, there exists $K''_\eta > 0$ such that

$$\int_0^t c(s)ds \le K''_\eta \left(1 + e^{K_0 t}\right), \qquad \forall t \ge 0. \tag{19}$$

Therefore, using (8) and integrating by parts we get, for every $T_1 > 0$,

$$\begin{aligned}\int_0^{T_1} e^{-\rho t} U_1(c(t))dt &\le C_1 \left(\frac{1}{\rho} + \int_0^{+\infty} e^{-\rho t} c(t)^{\beta_1} dt\right) \\ &= C_1 \left(\frac{1}{\rho} + \left[e^{-\rho t} \int_0^t c(s)^{\beta_1} ds\right]_0^{T_1} + \rho \int_0^{T_1} e^{-\rho t} \left(\int_0^t c(s)^{\beta_1} ds\right) dt\right)\end{aligned}$$

Therefore, the Jensen inequality yields

$$\begin{aligned}&\int_0^{T_1} e^{-\rho t} U_1(c(t))dt \\ &\qquad \le C_1 \left(\frac{1}{\rho} + \left(e^{-\rho T_1} T_1^{1-\beta_1} \left(\int_0^{T_1} c(s)ds\right)^{\beta_1}\right) + \rho \int_0^{T_1} e^{-\rho t} t^{1-\beta_1} \left(\int_0^t c(s)ds\right)^{\beta_1} dt\right).\end{aligned}$$

Now thanks to (19) and to assumption (11)

$$\lim_{T_1 \to +\infty} \left(e^{-\rho T_1} T_1^{1-\beta_1} \left(\int_0^{T_1} c(s)ds\right)^{\beta_1}\right) = 0$$

and the function

$$t \mapsto e^{-\rho t} t^{1-\beta_1} \left(\int_0^t c(s)ds\right)^{\beta_1}$$

is integrable on $[0, +\infty)$. So, letting $T_1 \to +\infty$ and using (19), we get

$$\int_0^{+\infty} e^{-\rho t} U_1(c(t))dt \le \frac{C_1}{\rho} + C_1 \rho \int_0^{+\infty} e^{-\rho t} t^{1-\beta_1} \left[K''_\eta (1 + e^{K_0 t})\right]^{\beta_1} dt. \tag{20}$$

Finally, invoking (18), (20) and (11) we complete the proof. □

2.1.4 Value function

For $\eta \in H_+$ the value function of our problem is defined as

$$V(\eta) := \sup_{c(\cdot) \in \mathcal{C}(\eta)} J(\eta, c(\cdot)), \tag{21}$$

with the agreement that $\sup \emptyset = -\infty$. Due to (15) we see that $V(\eta) < +\infty$ for every $\eta \in H_+$. The domain of V is defined as

$$\mathcal{D}(V) := \{\eta \in H_+ \mid V(\eta) > -\infty\}.$$

Proposition 2.11. *$H_{++} \subset \mathcal{D}(V)$ and $\mathcal{D}(V) = \{\eta \in H_+ \mid 0 \in \mathcal{C}(\eta)\}$.*

Proof. From Proposition 2.9-(2) we have $H_{++} \subset \{\eta \in H_+ \mid 0 \in \mathcal{C}(\eta)\}$. Thus we have to prove

$$\mathcal{D}(V) = \{\eta \in H_+ \mid 0 \in \mathcal{C}(\eta)\}.$$

If $\eta \in \mathcal{D}(V)$, then $\mathcal{C}(\eta) \neq \emptyset$. Thus, by Proposition 2.9-(1), we have $0 \in \mathcal{C}(\eta)$. This shows the inclusion

$$\mathcal{D}(V) \subset \{\eta \in H_+ \mid 0 \in \mathcal{C}(\eta)\}.$$

Let us prove the converse inclusion. To this end, take $\eta \in H_+$ such that $0 \in \mathcal{C}(\eta)$. Then there exists $\xi > 0$ such that

$$x(t;\eta,0) \geq \xi, \quad \forall t \in [0,T]. \tag{22}$$

Set $\eta^T = (\eta_0^T, \eta_1^T(\cdot))$, where

$$\eta_0^T := x(T;\eta,0), \quad \eta_1^T(s) := x(s+T;\eta,0), \quad s \in [-T,0].$$

By (22) we see that $\eta^T \in H_{++}$. Therefore, Proposition 2.9-(2) yields

$$x(t;\eta^T,0) \geq \eta_0^T e^{-C_{f_0}t}, \quad \forall t \geq 0. \tag{23}$$

By the semigroup property of the solution $x(\cdot;\eta,0)$ of (1) we have

$$x(t+T;\eta,0) = x(t;\eta^T,0), \quad \forall t \geq 0.$$

Combined with (23) the above equality yields

$$x(t;\eta,0) \geq \eta_0^T e^{-C_{f_0}(t-T)}, \quad \forall t \geq T. \tag{24}$$

Therefore, by (9) (see also Remark 2.5-(3)), we have that (22), (24) yield $J(\eta;0) > -\infty$, so $\eta \in \mathcal{D}(V)$. The proof is complete. □

Remark 2.12. It is straightforward to see that Proposition 2.11 above holds true if we replace assumption (9) with assumption (12). ■

Definition 2.13 (ε-optimal control)**.** *Let $\eta \in \mathcal{D}(V)$, $\varepsilon > 0$. An admissible control $c^\varepsilon(\cdot) \in \mathcal{C}(\eta)$ is said ε-optimal for the initial state η if $J(\eta; c^\varepsilon(\cdot)) > V(\eta) - \varepsilon$.*

Proposition 2.14. *The set $\mathcal{D}(V)$ is convex and the value function V is concave on $\mathcal{D}(V)$.*

Proof. Let $\eta, \bar\eta \in \mathcal{D}(V)$ and set, for $\lambda \in [0,1]$, $\eta_\lambda = \lambda\eta + (1-\lambda)\bar\eta$. For $\varepsilon > 0$, let $c^\varepsilon(\cdot) \in \mathcal{C}(\eta)$, $\bar c^\varepsilon(\cdot) \in \mathcal{C}(\bar\eta)$ be two ε-optimal controls for the initial states $\eta, \bar\eta$ respectively. Set $x(\cdot) = x(\cdot,\eta,c^\varepsilon(\cdot))$, $\bar x(\cdot) = x(\cdot;\eta,\bar c^\varepsilon(\cdot))$, $c^\lambda(\cdot) = \lambda c^\varepsilon(\cdot) + (1-\lambda)\bar c^\varepsilon(\cdot)$. Finally set $x_\lambda(\cdot) = \lambda x(\cdot) + (1-\lambda)\bar x(\cdot)$. We have

$$\begin{aligned}
x_\lambda'(t) &= \lambda x'(t) + (1-\lambda)\bar x'(t) \\
&= \lambda\left[rx(t) + f_0\left(x(t), \int_{-T}^0 a(\xi)x(t+\xi)d\xi\right) - c^\varepsilon(t)\right] \\
&\quad +(1-\lambda)\left[r\bar x(t) + f_0\left(\bar x(t), \int_{-T}^0 a(\xi)\bar x(t+\xi)d\xi\right) - \bar c^\varepsilon(t)\right] \\
&\leq rx_\lambda(t) + f_0\left(x_\lambda(t), \int_{-T}^0 a(\xi)x_\lambda(t+\xi)d\xi\right) - c^\lambda(t),
\end{aligned}$$

where the last inequality follows from the concavity of f_0. Let $x(\cdot;\eta_\lambda, c^\lambda(\cdot))$ be the solution of the equation

$$x'(t) = rx(t) + f_0\left(x(t), \int_{-T}^{0} a(\xi)x(t+\xi)d\xi\right) - c^\lambda(t),$$

with initial datum η_λ. Since $x_\lambda(\cdot) > 0$ by construction, by Lemma 2.8 we have $x(\cdot;\eta_\lambda, c^\lambda(\cdot)) \geq x_\lambda(\cdot) > 0$. This shows that $c^\lambda(\cdot) \in \mathcal{C}(\eta_\lambda)$. By concavity of U_1, U_2 and the monotonicity of U_2 we get

$$V(\eta_\lambda) \geq J(\eta_\lambda; c^\lambda(\cdot)) \geq \lambda J(\eta; c^\varepsilon(\cdot)) + (1-\lambda)J(\eta; \bar{c}^\varepsilon(\cdot)) > \lambda V(\eta) + (1-\lambda)V(\bar{\eta}) - \varepsilon.$$

Since ε is arbitrary, we conclude. □

By Hypothesis 2.5 (monotonicity of U_1, U_2) and by Lemma 2.8, we obtain the following result.

Proposition 2.15. *The function $\eta \mapsto V(\eta)$ is nondecreasing in the sense that*

$$\eta_0 \geq \bar{\eta}_0,\ \eta_1(\cdot) \geq \bar{\eta}_1(\cdot) \Longrightarrow V(\eta_0, \eta_1(\cdot)) \geq V(\bar{\eta}_0, \bar{\eta}_1(\cdot)).$$ □

We next show that V is strictly increasing with respect to the first variable (where it is finite).

Proposition 2.16. *Let $\bar{U}_1 = \lim_{c\to+\infty} U_1(c) \in [0, +\infty]$, $\bar{U}_2 = \lim_{x\to+\infty} U_2(x) \in (-\infty, +\infty]$. We have the following statements.*

1. *$V(\eta) < \frac{\bar{U}_1+\bar{U}_2}{\rho}$ for every $\eta \in H_+$.*
2. *$\lim_{\eta_0\to+\infty} V(\eta_0, \eta_1) = \frac{\bar{U}_1+\bar{U}_2}{\rho}$, for every $\eta_1 \in L^2_T$.*
3. *$V(\cdot, \eta_1(\cdot))$ is strictly increasing for every $\eta_1 \in L^2_T$ over $\{\eta_0 > 0 \mid (\eta_0, \eta_1(\cdot)) \in \mathcal{D}(V)\}$.*

Proof. 1. If $\bar{U}_1 = +\infty$ or $\bar{U}_2 = +\infty$, we have nothing to prove. Therefore assume that $\bar{U}_1, \bar{U}_2 < +\infty$ and let $\eta \in \mathcal{D}(V)$. From (19) we get

$$\int_0^1 c(\tau)d\tau \leq K, \quad \forall c(\cdot) \in \mathcal{C}(\eta),$$

where $K = K''_\eta(1+e^{K_0})$. Denoting by m the Lebesgue measure and defining

$$I_K := \{\tau \in [0,1] \mid c(\tau) \leq 2K\},$$

the above inequality implies $m(I_K) \geq 1/2$. Therefore

$$\begin{aligned}\int_0^{+\infty} e^{-\rho t}U_1(c(t))dt &\leq \int_{[0,1]\setminus I_K} e^{-\rho t}U_1(c(t))dt + \int_{I_K} e^{-\rho t}U_1(2K)dt + \int_1^{+\infty} e^{-\rho t}U_1(c(t))dt \\ &\leq \frac{\bar{U}_1}{\rho} - \int_{I_K} e^{-\rho t}\Big(\bar{U}_1 - U_1(2K)\Big)dt \leq \frac{\bar{U}_1}{\rho} - \int_{1/2}^{1} e^{-\rho t}\Big(\bar{U}_1 - U_1(2K)\Big)dt,\end{aligned}$$

where the last inequality follows from the fact that the function $t \mapsto e^{-\rho t}$ is decreasing. Since U_1 is strictly increasing (see Hypothesis 2.5), the quantity $\bar{U}_1 - U_1(2K)$ is strictly positive. Moreover, it does not depend on $c(\cdot)$. Since $\int_0^{+\infty} e^{-\rho t} U_2(x(t;\eta, c(\cdot)))dt \le \frac{\bar{U}_2}{\rho}$ for every $c(\cdot) \in \mathcal{C}(\eta)$, the claim is proved.

2. Let $K > 0$, $M > 0$ and let us define the control

$$c_{K,M}(t) := \begin{cases} M, & \text{if } t \in [0, K], \\ 0, & \text{if } t > K. \end{cases}$$

Given $\eta_1 \in L^2_T$, for any $\eta_0 > 0$ let us set $x_{K,M}(t;\eta_0) := x(t;(\eta_0,\eta_1), c_{K,M}(\cdot))$. Let

$$t_1(\eta_0, K, M) := \inf\{t \ge 0 \mid x_{K,M}(t, \eta_0) = 0\} > 0$$

and

$$q := \left(\sup_{\xi \in [-T,0]} a(\xi) \right) \left(\int_{-T}^0 \eta_1^-(\xi) d\xi \right).$$

Thanks to Hypothesis 2.2-(ii), f_0 is Lipschitz continuous and nondecreasing with respect to the second variable. Then there exists $C > 0$ such that $x_{K,M}(t;\eta_0)$ satisfies the differential inequality

$$x'_{K,M}(t;\eta_0) \ge -C(1 + x_{K,M}(t;\eta_0) + q) - M, \qquad \forall t \in [0, t_1(\eta_0, K, M)].$$

This actually shows that, for any $M > 0$, $K > 0$, $R > 0$, we can find η_0 such that $t_1(\eta_0, K, M) = +\infty$ (so $c_{K,M}(\cdot) \in \mathcal{C}(\eta_0, \eta_1(\cdot))$) and $x_{K,M}(\cdot;\eta_0) \ge R$ on $[0, K]$. By the arbitrariness of M, K, R the claim is proved.

3. We notice that, by item 2, the set $\{\eta_0 > 0 \mid (\eta_0, \eta_1(\cdot)) \in \mathcal{D}(V)\}$ is not empty for every $\eta_1 \in L^2_T$ and that, by definition of $\mathcal{D}(V)$, the function $V(\cdot, \eta_1(\cdot))$ is finite on this set. Fix $\eta_1(\cdot) \in L^2_T$. By Propositions 2.14, 2.15, we know that $\eta_0 \mapsto V(\eta_0, \eta_1(\cdot))$ is concave and nondecreasing. Then, assuming by contradiction that it is not strictly increasing on $\{\eta_0 > 0 \mid (\eta_0, \eta_1(\cdot)) \in \mathcal{D}(V)\}$, it should exist $\bar{\eta}_0 > 0$ such that $V(\cdot, \eta_1(\cdot))$ is constant on the half line $[\bar{\eta}_0, +\infty)$. This fact would contradict the first two claims of the present proposition, so we conclude. □

3 The delay problem rephrased in infinite dimension

Let $\hat{n} = (1, 0) \in H_+$ and let us consider, for $\eta \in H$ and $c(\cdot) \in L^1([0, +\infty); \mathbb{R}^+)$, the following evolution equation in the space H:

$$\begin{cases} X'(t) = AX(t) + F(X(t)) - c(t)\hat{n}, \\ X(0) = \eta. \end{cases} \tag{25}$$

In the equation above:

- $A : \mathcal{D}(A) \subset H \longrightarrow H$ is an unbounded operator defined by $A(\eta_0, \eta_1(\cdot)) := (r\eta_0, \eta_1'(\cdot))$ on
$$\mathcal{D}(A) := \{\eta \in H \mid \eta_1(\cdot) \in W^{1,2}_T,\ \eta_1(0) = \eta_0\};$$
- $F : H \longrightarrow H$ is a Lipschitz continuous map defined by
$$F(\eta_0, \eta_1(\cdot)) := (f(\eta_0, \eta_1(\cdot)), 0),$$
where $f(\eta_0, \eta_1(\cdot)) := f_0\left(\eta_0, \int_{-T}^0 a(\xi)\eta_1(\xi)d\xi\right)$.

It is well known that A is the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on H. If we intend η_1 extended to $[-r,+\infty)$ (defining η_1 as a whatever function on $(0,+\infty)$), then the explicit expression of $S(t)$ is

$$S(t)\eta = \left(\eta_0 e^{rt}, \mathbf{1}_{[-T,0]}(t+\cdot)\ \eta_1(t+\cdot) + \mathbf{1}_{[0,+\infty)}(t+\cdot)\ \eta_0 e^{r(t+\cdot)}\right).$$

Then we have

$$\begin{aligned}\|S(t)\eta\|^2 &\leq \left|\eta_0 e^{rt}\right|^2 + 2\int_{-T}^0 \left|\mathbf{1}_{[-T,0]}(t+\zeta)\ \eta_1(t+\zeta)\right|^2 d\zeta + 2\int_{-T}^0 \left|\mathbf{1}_{[0,+\infty)}(t+\zeta)\ \eta_0 e^{r(t+\zeta)}\right|^2 d\zeta \\ &\leq ((3+2T))e^{2rt}\|\eta\|^2.\end{aligned}$$

Therefore

$$\|S(t)\|_{\mathcal{L}(H)} \leq M e^{\omega t}, \tag{26}$$

where $M=(3+2T)$, $\omega = 2r$.

3.1 Mild solutions of the state equation

Definition 3.1. *A mild solution of (25) is a function $X \in C([0,+\infty);H)$ which satisfies the integral equation*

$$X(t) = S(t)\eta + \int_0^t S(t-\tau)F(X(\tau))d\tau + \int_0^t c(\tau)S(t-\tau)\hat{n}\, d\tau, \tag{27}$$

where both integrals above are understood as Bochner integrals of H-valued functions.

Theorem 3.2. *For any $\eta \in H$, there exists a unique mild solution of* (25).

Proof. Due to the Lipschitz continuity of F and to (26), the proof is a standard application of the fixed point theorem (see e.g. [9]). □

We denote by $X(\cdot;\eta,c(\cdot)) = (X_0(\cdot;\eta,c(\cdot)), X_1(\cdot;\eta,c(\cdot)))$ the unique solution to (25) for the initial state $\eta \in H$ and control $c(\cdot) \in L^1([0,+\infty);\mathbb{R}^+)$. The following equivalence result justifies our approach.

Proposition 3.3. *Let $\eta \in H$ and $c(\cdot) \in L^1([0,+\infty);\mathbb{R}^+)$. Let $x(\cdot)$, $X(\cdot)$ be respectively the unique solution to* (1) *and the unique mild solution to* (25) *starting from η and under the control $c(\cdot)$. Then, for any $t \geq 0$, we have the equality in H*

$$X(t) = \left(x(t), x(t+\xi)_{\xi\in[-T,0]}\right).$$

Proof. Let $x(\cdot)$ be a solution of (1) and let $Z(\cdot) = (x(\cdot), x(\cdot+\zeta)|_{\zeta\in[-T,0]})$. Then $Z(\cdot)$ belongs to the space $C([0,+\infty);H)$ because the function $[0,+\infty) \to \mathbb{R}$, $t \mapsto x(t)$ is (absolutely) continuous. Therefore, we have to prove that $Z(t) = (Z_0(t), Z_1(t))$ satisfies (25). The claim will follow by uniqueness.

For the first component we have to verify that

$$Z_0(t) = e^{rt}\eta_0 + \int_0^t e^{r(t-\tau)} f_0\left(Z_0(\tau), \int_{-T}^0 a(\xi)Z_1(\tau)(\xi)d\xi\right)d\tau - \int_0^t e^{r(t-\tau)}c(\tau)d\tau, \quad \forall t \geq 0.$$

This corrresponds to

$$x(t) = e^{rt}\eta_0 + \int_0^t e^{r(t-\tau)} f_0 \left(x(\tau), \int_{-T}^0 a(\xi)x(\tau+\xi)d\xi \right) d\tau - \int_0^t e^{r(t-\tau)} c(\tau)d\tau, \quad \forall t \geq 0,$$

This equality is true for every $t \geq 0$, since $x(\cdot)$ is a solution to (1).

Let us consider the second component. Taking into account the equality $\mathbf{1}_{[0,+\infty)}(t+\cdot-\tau) = \mathbf{1}_{[\tau,+\infty)}(t+\ \cdot)$, we see that have to verify the equality

$$\begin{aligned} Z_1(t)(\zeta) = \mathbf{1}_{[-T,0]}(t+\zeta)\eta_1(t+\zeta) + \mathbf{1}_{[0,+\infty)}(t+\zeta)\eta_0 e^{r(t+\zeta)} \\ + \int_0^t \mathbf{1}_{[\tau,+\infty)}(t+\zeta)\ e^{r(t+\zeta-\tau)} f_0 \left(Z_0(\tau), \int_{-T}^0 a(\xi) Z_1(\tau)(\xi) d\xi \right) d\tau \\ - \int_0^t \mathbf{1}_{[\tau,+\infty)}(t+\zeta)\ e^{r(t+\zeta-\tau)} c(\tau) d\tau, \quad \forall t \geq 0, \text{ for a.e. } \zeta \in [-T,0]. \end{aligned}$$

So, we have to verify the equality

$$\begin{aligned} x(t+\zeta) = \mathbf{1}_{[-T,0]}(t+\zeta)\eta_1(t+\zeta) + \mathbf{1}_{[0,+\infty)}(t+\zeta)\ \eta_0 e^{r(t+\zeta)} \\ + \int_0^t \mathbf{1}_{[\tau,+\infty)}(t+\zeta)\ e^{r(t+\zeta-\tau)} f_0 \left(x(\tau), \int_{-T}^0 a(\xi)x(\tau+\xi)d\xi \right) d\tau \\ - \int_0^t \mathbf{1}_{[\tau,+\infty)}(t+\zeta)\ e^{r(t+\zeta-\tau)} c(\tau) d\tau, \quad \forall t \geq 0, \text{ for a.e. } \zeta \in [-T,0]. \end{aligned} \tag{28}$$

Let $t \geq 0$. If $\zeta \in [-T,0]$ is such that $t+\zeta \in [-T,0]$, the equality (28) reduces to

$$x(t+\zeta) = \eta_1(t+\zeta).$$

This is true since η_1 is the initial condition of (1). If $\zeta \in [-T,0]$ is such that $t+\zeta \geq 0$, then (28) reduces to

$$x(t+\zeta) = \eta_0 e^{r(t+\zeta)} + \int_0^{t+\zeta} e^{r(t+\zeta-\tau)} f_0 \left(x(\tau), \int_{-T}^0 a(\xi)x(\tau+\xi)d\xi \right) d\tau - \int_0^{t+\zeta} e^{r(t+\zeta-\tau)} c(\tau)d\tau.$$

Setting $u = t+\zeta$ this equality becomes, for $u \geq 0$,

$$x(u) = \eta_0 e^{ru} + \int_0^t e^{r(u-\tau)} f_0 \left(x(\tau), \int_{-T}^0 a(\xi)x(\tau+\xi)d\xi \right) d\tau - \int_0^t e^{r(u-\tau)} c(\tau)d\tau.$$

Again this is true because $x(\cdot)$ solves (1). □

3.2 Continuity of the value function

We recall that the generator A of the semigroup $(S(t))_{t\geq 0}$ has bounded inverse in H given by

$$A^{-1}(\eta_0,\eta_1)(s) = \left(\frac{\eta_0}{r}, \frac{\eta_0}{r} - \int_s^0 \eta_1(\xi)d\xi \right), \quad s \in [-T,0].$$

It is well known that A^{-1} is compact in H. It is also clear that A^{-1} is an isomorphism of H onto $\mathcal{D}(A)$ endowed with the graph norm.

We define the $\|\cdot\|_{-1}$-norm on H by

$$\|\eta\|_{-1} := \|A^{-1}\eta\|.$$

In the next proposition we characterize the adjoint operator $A^\star$ and its domain $\mathcal{D}(A^\star)$.

Proposition 3.4. *Let $\eta = (\eta_0, \eta_1(\cdot)) \in H$. Then $\eta \in \mathcal{D}(A^\star)$ if and only if $\eta_1 \in W_T^{1,2}$ and $\eta_1(-T) = 0$. Moreover, if this is the case, then*

$$A^\star \eta = (r\eta_0 + \eta_1(0), -\eta_1'(\cdot)). \tag{29}$$

Proof. Let

$$(\eta_0, \eta_1) \in \mathcal{D} = \left\{\eta \in H : \eta_1 \in W_{-T}^{1,2},\ \eta_1(-T) = 0\right\}.$$

Then, for $\zeta \in \mathcal{D}(A)$,

$$\langle A\zeta, \eta\rangle = r\zeta_0\eta_0 + \int_{-T}^0 \zeta_1'(s)\eta_1(s)\,ds = r\zeta_0\eta_0 + \zeta_0\eta_1(0) - \int_{-T}^0 \zeta_1(s)\eta_1'(s)\,ds.$$

Thus $\zeta \mapsto \langle A\zeta, \eta\rangle$ is continuous on $\mathcal{D}(A)$ with respect to the norm $\|\cdot\|$. Therefore, $\eta \in D(A^\star)$ and

$$A^\star \eta = (r\eta_0 + \eta_1(0), -\eta_1'(\cdot)).$$

Therefore, $\eta \in \mathcal{D}(A^\star)$ and (29) holds. To show that $\mathcal{D}(A^\star) = \mathcal{D}$ note first that for $t \geq 0$

$$S^*(t)(\eta_0, \eta_1(\cdot)) = \left(e^{rt}\left(\eta_0 + \int_{(-t)\vee(-T)}^0 \eta_1(\xi)e^{r\xi}d\xi\right), \eta_1(\cdot - t)\mathbf{1}_{[-T,0]}(\cdot - t)\right). \tag{30}$$

Clearly, $\mathcal{D}$ is dense in H and it is easy to check that $S^*(t)\mathcal{D} \subset \mathcal{D}$ for any $t \geq 0$. Hence, by Theorem 1.9 on p. 8 of [14], $\mathcal{D}$ is dense in $\mathcal{D}(A^\star)$ endowed with the graph norm. Finally, using (29) it is easy to show that $\mathcal{D}$ is closed in the graph norm of $A^\star$ and therefore $\mathcal{D}(A^\star) = \mathcal{D}$. □

Lemma 3.5. *The map F is Lipschitz continuous with respect to the norm $\|\cdot\|_{-1}$.*

Proof. Due to the Lipschitz continuity of f_0, it suffices to prove that

$$|\eta_0| + \left|\int_{-T}^0 a(\xi)\eta_1(\xi)d\xi\right| \leq C_{a(\cdot)}\|\eta\|_{-1}, \quad \forall \eta \in H. \tag{31}$$

Indeed, since $|\eta_0| \leq r\|\eta\|_{-1}$ $(0, a(\cdot)) \in \mathcal{D}(A^\star)$, we find that

$$\begin{aligned}\left|\int_{-T}^0 a(\xi)\eta_1(\xi)d\xi\right| &= |\langle(0,a(\cdot)),\eta\rangle| = |\langle(0,a(\cdot)), AA^{-1}\eta\rangle| \\ &= |\langle A^\star(0,a(\cdot)), A^{-1}\eta\rangle| \leq \|A^\star(0,a(\cdot))\| \cdot \|\eta\|_{-1}.\end{aligned}$$

So, since $|\eta_0| \leq r\|\eta\|_{-1}$, we get (31) with $C_{a(\cdot)} = r + \|A^\star(0,a(\cdot))\|$. □

Remark 3.6. The condition $a(-T) = 0$ is needed to get the previous result. Indeed, consider for example the case $a(\cdot) \equiv 1$. Then the sequence

$$\eta^n = (\eta_0^n, \eta_1^n(\cdot)), \quad \eta_0^n := 0,\ \eta_1^n(\cdot) := \mathbf{1}_{[-T,-T+1/n]}(\cdot), \quad n \geq 1,$$

is such that

$$\left|\int_{-T}^0 a(\xi)\eta_1^n(\xi)d\xi\right| = 1 \ \ \forall n \geq 1, \qquad \|\eta^n\|_{-1} \to 0 \text{ when } n \to \infty,$$

so that (31) cannot be satisfied. If for example $f_0(r,u) = u$, the result does not hold. ■

Lemma 3.7. *Let $X(\cdot), \bar{X}(\cdot)$ be the mild solutions to (25) starting respectively from $\eta, \bar{\eta} \in H$ and both under the null control. Then there exists a constant $C > 0$ such that*

$$\|X(t) - \bar{X}(t)\|_{-1} \le C\|\eta - \bar{\eta}\|_{-1}, \quad \forall t \in [0,T].$$

In particular

$$|X_0(t) - \bar{X}_0(t)| \le rC\|\eta - \bar{\eta}\|_{-1}, \quad \forall t \in [0,T].$$

Proof. Invoking (27) we obtain, for all $t \in [0,T]$,

$$X(t) - \bar{X}(t) = S(t)(\eta - \bar{\eta}) + \int_0^t S(t-\tau)\left[F(X(\tau)) - F(\bar{X}(\tau))\right] d\tau.$$

Hence,

$$A^{-1}(X(t) - \bar{X}(t)) = S(t)A^{-1}(\eta - \bar{\eta}) + \int_0^t S(t-\tau)\, A^{-1}\left[F(X(\tau)) - F(\bar{X}(\tau))\right] d\tau.$$

Therefore, taking into account Lemma 3.5, there exists some $K > 0$ such that

$$\|X(t) - \bar{X}(t)\|_{-1} \le K\left(\|\eta - \bar{\eta}\|_{-1} + \int_0^t \|X(\tau) - \bar{X}(\tau)\|_{-1} d\tau\right).$$

The claim follows by Gronwall's Lemma. □

Proposition 3.8.

1. *The set $\mathcal{D}(V)$ is open in the space $(H, \|\cdot\|_{-1})$.*
2. *V is continuous with respect to $\|\cdot\|_{-1}$ on $\mathcal{D}(V)$. Moreover*

$$(\eta_n) \subset \mathcal{D}(V), \quad \eta_n \rightharpoonup \eta \in \mathcal{D}(V) \quad \Longrightarrow \quad V(\eta_n) \to V(\eta). \tag{32}$$

Proof. 1. Let $\bar{\eta} \in \mathcal{D}(V)$. We will show that

$$\exists \varepsilon > 0, \exists M > 0 \;\text{ such that }\; V(\eta) \ge -M, \quad \forall \eta \in B_{-1}(\bar{\eta}, \varepsilon) := \{\eta \in H_+ \mid \|\eta - \bar{\eta}\|_{-1} < \varepsilon\}. \tag{33}$$

In particular (33) implies that $\mathcal{D}(V)$ is $\|\cdot\|_{-1}$-open. Let $\eta \in H_+$ and set

$$\bar{X}(\cdot) := X(\cdot; \bar{\eta}, 0), \quad X(\cdot) := X(\cdot; \eta, 0).$$

By Proposition 2.9-(2) there exists $\xi > 0$ such that

$$\bar{X}_0(t) \ge \xi, \quad \forall t \in [0,T].$$

Let C be the constant of Lemma 3.7 and take $\varepsilon \in \left(0, \xi(2rC)^{-1}\right)$. By Lemma 3.7, we get for any η such that $\|\eta - \bar{\eta}\|_{-1} < \varepsilon$

$$X_0(t) \ge \xi/2, \quad \forall t \in [0,T]. \tag{34}$$

Observe that, by Proposition 3.3, $X_0(t) = x(t; \eta, 0)$, where $x(\cdot; \eta, 0)$ is the solution of (1). Then, arguing as in the second part of the proof of Proposition 2.11, from (34) we get

$$X_0(t) \ge X_0(T)e^{-C_{f_0}(t-T)} \ge \frac{\xi}{2}e^{-C_{f_0}(t-T)}, \quad \forall t \ge T. \tag{35}$$

Thanks to (9) (see also Remark 2.5-(3)), the above inequality and (34) show that there exists $M > 0$ such that $J(\eta; 0) \geq -M$ for every $\eta \in B_{-1}(\bar{\eta}, \varepsilon)$. Therefore also $V(\eta) \geq -M$ for every $\eta \in B_{-1}(\bar{\eta}, \varepsilon)$ and (33) is proved.

2. From (33) it follows that V is $\|\cdot\|_{-1}$-locally bounded from below at the points of $\mathcal{D}(V)$. As V is also concave (Proposition 2.14), the $\|\cdot\|_{-1}$-continuity of V on $\mathcal{D}(V)$ follows by a classical result of Convex Analysis (see e.g. [17], Chapter 1, Corollary 2.4). The claim (32) follows from the first claim, since A^{-1} is compact. □

Remark 3.9. $\mathcal{D}(V)$ is open also with respect to $\|\cdot\|$. ■

Remark 3.10. It is straightforward to see that Proposition 3.8 above holds even if we replace assumption (9) with assumption (12). ■

3.3 Properties of superdifferential

Recall that, if $O \subset H$ is open and $v : O \to \mathbb{R}$ is continuous, the subdifferential and the superdifferential of v at a point $\eta \in O$ are the sets

$$D^{-}v(\eta) := \left\{ \alpha \in H \;\middle|\; \liminf_{\zeta \to \eta} \frac{v(\zeta) - v(\eta) - \langle \zeta - \eta, \alpha \rangle}{\|\zeta - \eta\|} \geq 0 \right\},$$

$$D^{+}v(\eta) := \left\{ \alpha \in H \;\middle|\; \limsup_{\zeta \to \eta} \frac{v(\zeta) - v(\eta) - \langle \zeta - \eta, \alpha \rangle}{\|\zeta - \eta\|} \leq 0 \right\}.$$

The set of the "reachable gradients" at $\eta \in O$ is defined as

$$D^{*}v(\eta) := \left\{ \alpha \in H \;\middle|\; \exists \eta_n \to \eta,\ \eta_n \in O, \text{ such that } \exists \nabla v(\eta_n) \text{ and } \nabla v(\eta_n) \to \alpha \right\}.$$

Consider now the case when v is concave (and continuous). In this case $D^{+}v$ is not empty at every point of O, bounded, weakly closed (see [28], Chapter 1, Proposition 1.11) and

$$D^{+}v(\eta) = \left\{ \alpha \in H \;\middle|\; v(\zeta) - v(\eta) \leq \langle \zeta - \eta, \alpha \rangle, \quad \forall \zeta \in O \right\}.$$

Moreover the set-valued map $O \to \mathcal{P}(H)$, $\eta \mapsto D^{+}v(\eta)$ is locally bounded (see again [28], Chapter 1, Proposition 1.11). Also we have the useful representation (see [29], pp.319-320)

$$D^{+}v(\eta) = \overline{co}\left(D^{*}v(\eta)\right), \quad \eta \in O. \tag{36}$$

Defining the directional superdifferential of v along $\hat{n} = (1, 0) \in H$ at $\eta \in O$ as

$$D^{+}_{\hat{n}}v(\eta) = \{\alpha_0 \in \mathbb{R} \mid v(\zeta_0, \eta_1) - v(\eta_0, \eta_1) \leq \alpha_0(\zeta_0 - \eta_0), \quad \forall \zeta_0 \in \mathbb{R} \text{ s.t. } (\zeta_0, \eta_1) \in O\},$$

we have that this set is a nonempty closed and bounded interval $[a, b] \subset \mathbb{R}$. More precisely, since $v(\cdot, \eta_1)$ is concave, we have

$$a = v^{+}_{\eta_0}(\eta), \quad b = v^{-}_{\eta_0}(\eta),$$

where $v^{+}_{\eta_0}(\eta), v^{-}_{\eta_0}(\eta)$ denote respectively the right and the left derivatives of the function $v(\cdot, \eta_1)$ at the point η_0. By definition of $D^{+}v(\eta)$, the projection of $D^{+}v(\eta)$ on $\hat{n}$ must be contained in $D^{+}_{\hat{n}}v(\eta)$, that is

$$D^{+}_{\hat{n}}v(\eta) \supset \{\alpha_0 \mid \alpha \in D^{+}v(\eta)\}. \tag{37}$$

On the other hand, Proposition 2.24 in [28], Chapter 1, states that

$$a = \inf\{\langle \alpha, \hat{n}\rangle \mid \alpha \in D^+v(\eta)\}, \quad b = \sup\{\langle \beta, \hat{n}\rangle \mid \beta \in D^+v(\eta)\}.$$

and that the sup and inf above are attained. This means that there exists $\alpha, \beta \in D^+v(\eta)$ such that

$$a = \alpha_0 = \langle \alpha, \hat{n}\rangle, \quad b = \beta_0 = \langle \beta, \hat{n}\rangle.$$

Since $D^+v(\eta)$ is convex, we see that also the converse inclusion of (37) is true. Therefore

$$D^+_{\hat{n}}v(\eta) = \{\alpha_0 \mid \alpha \in D^+v(\eta)\}. \tag{38}$$

Lemma 3.11. *The following statements hold:*

1. $A^{-1}\mathcal{D}(V)$ *is a convex open set in* $(\mathcal{D}(A), \|\cdot\|)$.
2. $\mathcal{O} := Int_{(H,\|\cdot\|)}\left(Clos_{(H,\|\cdot\|)}\left(A^{-1}\mathcal{D}(V)\right)\right)$ *is a convex open set of* $(H, \|\cdot\|)$.
3. $A^{-1}\mathcal{D}(V) \subset \mathcal{O}$ *and* $A^{-1}\mathcal{D}(V)$ *is dense in* $\mathcal{O}$.

Proof. 1. Observe first that, since A^{-1} is one-to-one, there is a one-to-one correspondence between the elements $\eta \in H$ and $p \in \mathcal{D}(A)$. For every $\eta, \bar{\eta} \in H$, we set $p = A^{-1}\eta$, $\bar{p} = A^{-1}\bar{\eta}$. Given $\varepsilon > 0$ we have

$$\begin{aligned}\{p \in \mathcal{D}(A) \mid \|p - \bar{p}\| < \varepsilon\} &= \{A^{-1}\eta,\ \eta \in H \mid \|\eta - \bar{\eta}\|_{-1} < \varepsilon\} \\ &= A^{-1}\{\eta \in H \mid \|\eta - \bar{\eta}\|_{-1} < \varepsilon\}.\end{aligned} \tag{39}$$

Recall that $\mathcal{D}(V)$ is open in $(H, \|\cdot\|_{-1})$ by Proposition 3.8-(1). Therefore, given $\bar{\eta} \in \mathcal{D}(V)$, we may find $\varepsilon > 0$ such that

$$\{\eta \in H \mid \|\eta - \bar{\eta}\|_{-1} < \varepsilon\} \subset \mathcal{D}(V). \tag{40}$$

Let $\bar{p} \in \mathcal{D}(V)$. We have by (39) and (40)

$$\{p \in \mathcal{D}(A) \mid \|p - \bar{p}\| < \varepsilon\} \subset A^{-1}\mathcal{D}(V),$$

which shows the claim.

2. The set $\mathcal{O}$ is open by definition. Since A^{-1} is linear and $\mathcal{D}(V)$ is convex, the fact that $\mathcal{O}$ is convex follows from the fact that the interior and the closure of convex sets are convex.

3. The fact that $A^{-1}\mathcal{D}(V)$ is dense in $\mathcal{O}$ follows from its definition. Let us prove the inclusion $A^{-1}\mathcal{D}(V) \subset \mathcal{O}$. To this aim take $\bar{p} \in A^{-1}\mathcal{D}(V)$. We must prove that there exists $\varepsilon > 0$ such that

$$B^H_\varepsilon := \{p \in H \mid \|p - \bar{p}\| < \varepsilon\} \subset \text{Clos}_{(H,\|\cdot\|)}\left(A^{-1}\mathcal{D}(V)\right). \tag{41}$$

Since $A^{-1}\mathcal{D}(V)$ is open in $(\mathcal{D}(A), \|\cdot\|)$ (point 1), we may find $\varepsilon_0 > 0$ such that

$$B^{\mathcal{D}(A)}_{\varepsilon_0} := \{p \in \mathcal{D}(A) \mid \|p - \bar{p}\| < \varepsilon_0\} \subset A^{-1}\mathcal{D}(V) \subset \text{Clos}_{(H,\|\cdot\|)}\left(A^{-1}\mathcal{D}(V)\right). \tag{42}$$

Take $\varepsilon = \varepsilon_0/2$ in (41). If $p \in B^H_\varepsilon$ is such that $p \in \mathcal{D}(A)$, then $p \in \text{Clos}_{(H,\|\cdot\|)}\left(A^{-1}\mathcal{D}(V)\right)$ by (42). If $p \in B^H_\varepsilon$ is such that $p \notin \mathcal{D}(A)$, we may find a sequence $(p_n) \subset \mathcal{D}(A)$ such that $\|p_n - p\| < \varepsilon$ and $p_n \to p$. Then $p_n \in B^{\mathcal{D}(A)}_{\varepsilon_0}$, so $p \in \text{Clos}_{(H,\|\cdot\|)}\big(B^{\mathcal{D}(A)}_{\varepsilon_0}\big)$. Again by (42) we get $p \in \text{Clos}_{(H,\|\cdot\|)}\big(A^{-1}\mathcal{D}(V)\big)$. $\square$

Proposition 3.12. *Let $v : \mathcal{D}(V) \to \mathbb{R}$ be a concave function continuous with respect to $\|\cdot\|_{-1}$. Then*

1. *$v = u \circ A^{-1}$, where $u : \mathcal{O} \subset H \to \mathbb{R}$ is a concave $\|\cdot\|$-continuous function on the open set $\mathcal{O}$ defined in Lemma 3.11-(2).*

2. *$D^+v(\eta) \subset \mathcal{D}(A^\star)$, for any $\eta \in \mathcal{D}(V)$.*

3. *$D^+u(A^{-1}\eta) = A^\star D^+v(\eta)$, for any $\eta \in \mathcal{D}(V)$. In particular, since $A^\star$ is injective, v is differentiable at η if and only if u is differentiable at $A^{-1}\eta$. In this case $\nabla u(A^{-1}\eta) = A^\star \nabla v(\eta)$.*

4. *Let $\eta \in \mathcal{D}(V)$. If $\alpha \in D^*v(\eta)$, then $\alpha \in \mathcal{D}(A^\star)$ and there exists a sequence $\eta_n \to \eta$ such that*
$$\exists \nabla v(\eta_n),\ \forall n \in \mathbb{N}, \quad \text{and} \quad \nabla v(\eta_n) \to \alpha,\ A^\star \nabla v(\eta_n) \rightharpoonup A^\star \alpha.$$

Proof. Observe first that, since A^{-1} is one-to-one, there is a one-to-one correspondence between the elements $\eta \in \mathcal{D}(V)$ and $p \in A^{-1}\mathcal{D}(V)$.

1. Let us define the function $u_0 : A^{-1}\mathcal{D}(V) \to \mathbb{R}$ by
$$u_0(p) := v(Ap).$$
Since v is concave and continuous, we see that u_0 is a concave continuous function on $(A^{-1}\mathcal{D}(V), \|\cdot\|)$ too. Since u_0 is concave, it is locally Lipschitz continuous on $(A^{-1}(\mathcal{D}(V), \|\cdot\|)$. Moreover, by Lemma 3.11-(3), $A^{-1}\mathcal{D}(V)$ is $\|\cdot\|$-dense in $\mathcal{O}$. So u_0 can be extended to a concave $\|\cdot\|$-continuous function u defined on $\mathcal{O}$. This function satisfies the claim by construction.

2. Let $\eta \in \mathcal{D}(V)$, $\alpha \in D^+v(\eta)$. Then
$$v(\zeta) - v(\eta) \leq \langle \zeta - \eta, \alpha \rangle, \qquad \forall \zeta \in \mathcal{D}(V).$$
So, setting $p = A^{-1}\eta$, $q = A^{-1}\zeta$,
$$u(q) - u(p) \leq \langle A(q-p), \alpha \rangle, \qquad \forall q \in A^{-1}\mathcal{D}(V).$$
Hence, the function $(\mathcal{D}(A), \|\cdot\|) \longrightarrow \mathbb{R}$, $q \longmapsto \langle Aq, \alpha \rangle$, is lower semicontinuous at p. It is also linear and therefore it is continuous on $(\mathcal{D}(A), \|\cdot\|)$. So, we conclude that $\alpha \in \mathcal{D}(A^\star)$.

3. Let $\eta \in \mathcal{D}(V)$, $\alpha \in D^+v(\eta)$. Then
$$v(\zeta) - v(\eta) \leq \langle \zeta - \eta, \alpha \rangle, \qquad \forall \zeta \in \mathcal{D}(V).$$
Thus, setting $p = A^{-1}\eta$, $q = A^{-1}\zeta$,
$$u(q) - u(p) \leq \langle A(q-p), \alpha \rangle = \langle q - p, A^\star \alpha \rangle, \qquad \forall q \in A^{-1}\mathcal{D}(V).$$
So, $A^\star\alpha \in D^+u(p)$. This proves the inclusion $D^+u(A^{-1}\eta) \supset A^\star D^+v(\eta)$.

Conversely let $p \in A^{-1}(\mathcal{D}(V))$ and $w \in D^+u(p)$. Then
$$u(q) - u(p) \leq \langle q - p, w \rangle, \quad \forall q \in A^{-1}\mathcal{D}(V).$$

Thus, setting $\eta = Ap$, $\zeta = Aq$,

$$v(\zeta) - v(\eta) \leq \langle A^{-1}(\zeta - \eta), w \rangle = \langle \zeta - \eta, (A^{-1})^\star w \rangle, \qquad \forall \zeta \in \mathcal{D}(V).$$

Since $(A^{-1})^\star = (A^\star)^{-1}$, we get $(A^\star)^{-1} w \in D^+ v(\bar{\eta})$. This proves the inclusion $D^+ u(A^{-1}\eta) \subset A^\star D^+ v(\eta)$.

4. Let $\eta \in \mathcal{D}(V)$ and $\alpha \in D^* v(\eta)$. By definition of $D^* v(\eta)$, we can find a sequence $(\eta_n) \subset \mathcal{D}(V)$ such that $\eta_n \to \eta$, $\nabla v(\eta_n)$ exists for any $n \in \mathbb{N}$ and $\nabla v(\eta_n) \to \alpha$. Setting $p_n = A^{-1}\eta_n$, thanks to claim 3 also $\nabla u(p_n)$ exists and $\nabla u(p_n) = A^\star \nabla v(\eta_n)$. Since u is concave, the set-valued map $p \mapsto D^+ u(p)$ is locally bounded. Hence the sequence $\nabla u(p_n)$ is bounded. Therefore from any subsequence we can extract a subsubsequence weakly converging to some element $j \in H$. The operator $A^\star$ is closed, so the graph of A^* is closed in $(H \times H, \|\cdot\| \times \|\cdot\|)$. Such graph is a convex set, so it is also closed in $(H \times H, \|\cdot\| \times \mathcal{T}_w)$, where $\mathcal{T}_w$ is the weak topology of H. Therefore we can say that $\alpha \in \mathcal{D}(A^\star)$ and $j = A^\star \alpha$. Since this holds for any subsequence, we conclude that $A^\star \nabla v(\eta_n) = \nabla u(p_n) \rightharpoonup A^\star \alpha$. □

4 Dynamic Programming

In this section we consider the Dynamic Programming that in our case can be stated as follows.

Theorem 4.1 (Dynamic Programming Principle)**.** *For any $\eta \in \mathcal{D}(V)$ and for any $s \geq 0$,*

$$V(\eta) = \sup_{c(\cdot) \in \mathcal{C}(\eta)} \left[\int_0^s e^{-\rho t} \left(U_1(c(t) + U_2(X_0(t)) \right) dt + e^{-\rho s} V(X(s)) \right],$$

where $X(\cdot) := X(\cdot; \eta, c(\cdot))$.

Proof. See e.g. [27], Theorem 1.1 in Chapter 6. The proof can be easily adapted to our constrained case. □

The differential version of the Dynamic Programming Principle is the Hamilton-Jacobi-Bellman (from now on HJB) equation, which in our case reads as

$$\rho v(\eta) = \langle \eta, A^\star \nabla v(\eta) \rangle + f(\eta) v_{\eta_0}(\eta) + U_2(\eta_0) + \mathcal{H}(v_{\eta_0}(\eta)), \quad \eta \in \mathcal{D}(V), \tag{43}$$

where $\mathcal{H}$ is the Legendre transform of U_1, i.e.

$$\mathcal{H}(\alpha_0) := \sup_{c \geq 0} \left(U_1(c) - \alpha_0 c \right), \quad \alpha_0 \in \mathbb{R}.$$

Due to Hyphothesis 2.5-(i) and to Corollary 26.4.1 of [30], $\mathcal{H}$ is finite and strictly convex on $(0, +\infty)$. Notice that, thanks to Proposition 2.16-(3),

$$D^+_{\hat{n}} V(\eta) \subset (0, \infty), \quad \forall \eta \in \mathcal{D}(V),$$

where $\hat{n} = (1, 0) \in H$.

4.1 Viscosity solutions

We now prove that the value function V is a viscosity solution of the HJB equation (43). To this end we define the following set of test functions

$$\tau := \left\{ \varphi \in C^1(H) \mid \nabla\varphi(\cdot) \in \mathcal{D}(A^\star),\ A^\star\nabla\varphi : H \to H \text{ is continuous} \right\}. \tag{44}$$

Let us define, for $c \geq 0$, the operator $\mathcal{L}^c$ on τ by

$$[\mathcal{L}^c\varphi](\eta) := -\rho\varphi(\eta) + \langle \eta, A^\star\nabla\varphi(\eta)\rangle + f(\eta)\varphi_{\eta_0}(\eta) - c\varphi_{\eta_0}(\eta).$$

Lemma 4.2. *Let $\varphi \in \tau$, $c(\cdot) \in L^1([0,+\infty);\mathbb{R}^+)$ and set $X(t) := X(t;\eta,c(\cdot))$. The following identity holds for any $t \geq 0$:*

$$e^{-\rho t}\varphi(X(t)) - \varphi(\eta) = \int_0^t e^{-\rho s}[\mathcal{L}^{c(s)}\varphi](X(s))ds.$$

Proof. The statement holds if we replace A with the Yoshida approximations. Then we can pass to the limit and get the claim thanks to the regularity properties of the functions belonging to τ. See also [27], Chapter 2, Proposition 5.5. □

Definition 4.3. *(i) A continuous function $v : \mathcal{D}(V) \to \mathbb{R}$ is called a viscosity subsolution of* (43) *on $\mathcal{D}(V)$ if for any $\varphi \in \tau$ and any $\eta_M \in \mathcal{D}(V)$ such that $v - \varphi$ has a $\|\cdot\|$-local maximum at η_M we have*

$$\rho v(\eta_M) \leq \langle \eta_M, A^\star\nabla\varphi(\eta_M)\rangle + f(\eta_M)\varphi_{\eta_0}(\eta_M) + U_2(\eta_0) + \mathcal{H}(\varphi_{\eta_0}(\eta_M)).$$

(ii) A continuous function $v : \mathcal{D}(V) \to \mathbb{R}$ is called a viscosity supersolution of (43) *on $\mathcal{D}(V)$ if for any $\varphi \in \tau$ and any $\eta_m \in \mathcal{D}(V)$ such that $v - \varphi$ has a $\|\cdot\|$-local minimum at η_m we have*

$$\rho v(\eta_m) \geq \langle \eta_m, A^\star\nabla\varphi(\eta_m)\rangle + f(\eta_m)\varphi_{\eta_0}(\eta_m) + U_2(\eta_0) + \mathcal{H}(\varphi_{\eta_0}(\eta_m)).$$

(iii) A continuous function $v : \mathcal{D}(V) \to \mathbb{R}$ is called a viscosity solution of (43) *on $\mathcal{D}(V)$ if it is both a viscosity sub and supersolution.*

Theorem 4.4. *The value function V is a viscosity solution to* (43) *on $\mathcal{D}(V)$.*

Proof. (i) We prove that V is a viscosity subsolution. Let $(\eta_M, \varphi) \in \mathcal{D}(V) \times \tau$ be such that $V - \varphi$ has a local maximum at η_M. Without loss of generality we can suppose $V(\eta_M) = \varphi(\eta_M)$. Let us suppose, by contradiction that there exists $\nu > 0$ such that

$$2\nu \leq \rho V(\eta_M) - \big(\langle \eta_M, A^\star\nabla\varphi(\eta_M)\rangle + f(\eta_M)\varphi_{\eta_0}(\eta_M) + U_2(\eta_0) + \mathcal{H}(\varphi_{\eta_0}(\eta_M))\big).$$

Let us define the function

$$\tilde{\varphi}(\eta) := V(\eta_M) + \langle \nabla\varphi(\eta_M), \eta - \eta_M\rangle + \|\eta - \eta_M\|_{-1}^2.$$

We have

$$\nabla\tilde{\varphi}(\eta) = \nabla\varphi(\eta_M) + (A^\star)^{-1}A^{-1}(\eta - \eta_M),$$

Thus $\tilde{\varphi}$ is a test function and we must have also

$$2\nu \leq \rho V(\eta_M) - \big(\langle \eta_M, A^\star\nabla\tilde{\varphi}(\eta_M)\rangle + f(\eta_M)\tilde{\varphi}_{\eta_0}(\eta_M) + U_2(\eta_0) + \mathcal{H}(\tilde{\varphi}_{\eta_0}(\eta_M))\big).$$

By concavity of V we have

$$V(\eta_M) = \tilde{\varphi}(\eta_M), \qquad \tilde{\varphi}(\eta) \geq V(\eta) + \|\eta - \eta_M\|_{-1}^2, \ \eta \in \mathcal{D}(V). \tag{45}$$

By the continuity property of $\tilde{\varphi}$ we can find $\varepsilon > 0$ such that

$$\nu \leq \rho V(\eta) - \left(\langle \eta, A^\star \nabla \tilde{\varphi}(\eta)\rangle + f(\eta)\tilde{\varphi}_{\eta_0}(\eta) + U_2(\eta_0) + \mathcal{H}(\tilde{\varphi}_{\eta_0}(\eta))\right), \quad \eta \in B(\eta_M, \varepsilon). \tag{46}$$

Take a sequence $\delta_n > 0$, $\delta_n \to 0$ and, for any n, take a δ_n-optimal control $c_n(\cdot) \in \mathcal{C}(\eta_M)$. Set $X^n(\cdot) := X(\cdot; \eta_M, c_n(\cdot))$ and define

$$t_n := \inf\{t \geq 0 \mid \|X^n(t) - \eta_M\| = \varepsilon\} \wedge 1,$$

with the agreement that $\inf \emptyset = +\infty$. Then t_n belongs to $(0, 1]$. Moreover, by continuity of trajectories, $X^n(t) \in B(\eta_M, \varepsilon)$, for $t \in [0, t_n)$. Using the δ_n-optimality of $c_n(\cdot)$, (45) and (46), we would arrive to write

$$\delta_n \geq t_n \nu + e^{-\rho t_n} \|X^n(t_n) - \eta_M\|_{-1}^2, \tag{47}$$

We distinguish two cases:

$$\limsup_n t_n = 0, \qquad \text{or} \qquad \limsup_n t_n > 0.$$

In the first case it must be

$$\|X^n(t_n) - \eta_M\|_{-1}^2 \to 0.$$

Let us show that this is impossible. The above limit implies in particular that

$$|X_0^n(t_n) - (\eta_M)_0| \to 0. \tag{48}$$

Moreover, by definition of t_n, it has to be

$$|X_0^n(t) - (\eta_M)_0| \leq \varepsilon, \quad t \in [0, t_n]. \tag{49}$$

Since $t_n \to 0$, taking into account (49), we have also

$$\|X_1^n(t_n) - (\eta_M)_1\|_{L_T^2} \to 0. \tag{50}$$

The limits (48) and (50) are not compatible with the definition of t_n and the contradiction arises. In the second case, we get from (47) that $\delta_n \geq t_n \nu$. We can suppose, eventually passing to a subsequence, that $t_n \to \bar{t} \in (0, 1]$. Since $\delta_n \to 0$ and $t_n \nu \to \bar{t}\nu$, again the contradiction arises.

(ii) The proof that V is a viscosity supersolution follows the same line and it is indeed easier. See e.g. [27], Theorem 3.2, Chapter 6. □

4.2 Smoothness of viscosity solutions

We start with a lemma.

Lemma 4.5. *Let $v : \mathcal{D}(V) \to \mathbb{R}$ be a concave $\|\cdot\|_{-1}$-continuous function. Assume that $\eta \in \mathcal{D}(V)$ is a point of differentiability for v and that $\nabla v(\eta) = \alpha$. Then*

1. *There exists a test function φ such that $v - \varphi$ has a local maximum at η and $\nabla \varphi(\eta) = \alpha$.*

2. There exists a test function φ such that $v-\varphi$ has a local minimum at η and $\nabla\varphi(\eta)=\alpha$.

Proof. Thanks to Proposition 3.12-(2) and to the concavity of v, the first statement is clearly satisfied by the function $\langle\cdot,\alpha\rangle$. We prove now the second statement, which is more delicate. Let u be defined as in Proposition 3.12-(1). The first and third claim of Proposition 3.12 yield that $\alpha\in\mathcal{D}(A^\star)$, u is differentiable at $p:=A^{-1}\eta$ and $\nabla u(p)=A^\star\alpha$. This yields

$$u(q)-u(p)-\langle q-p,A^\star\alpha\rangle\geq-\|q-p\|\cdot\varepsilon(\|q-p\|),\quad \forall q\in A^{-1}\mathcal{D}(V),$$

for some $\varepsilon:[0,+\infty)\to[0,+\infty)$ increasing and such that $\varepsilon(r)\to 0$, when $r\to 0$. The previous inequality can be rewritten as

$$u(q)-u(p)-\langle A(q-p),\alpha\rangle\geq-\|q-p\|\cdot\varepsilon(\|q-p\|),\quad \forall q\in A^{-1}\mathcal{D}(V).$$

Therefore, defining $\zeta=Aq$ for $q\in A^{-1}\mathcal{D}(V)$ and recalling that A is one-to-one from $A^{-1}\mathcal{D}(V)$ to $\mathcal{D}(V)$,

$$v(\zeta)-v(\eta)-\langle\zeta-\eta,\alpha\rangle\geq-\|\zeta-\eta\|_{-1}\cdot\varepsilon\left(\|\zeta-\eta\|_{-1}\right),\quad\forall\zeta\in\mathcal{D}(V). \tag{51}$$

We look for a test function of this form:

$$\varphi(\zeta)=v(\eta)+\langle\zeta-\eta,\alpha\rangle-g\left(\|\zeta-\eta\|_{-1}\right),\quad \zeta\in\mathcal{D}(V),$$

where $g:[0,+\infty)\to[0,+\infty)$ is a suitable increasing C^1 function such that $g(0)=g'(0)=0$. Notice that, since $g(0)=0$, we have $\varphi(\eta)=v(\eta)$. So, in order to prove that $v-\varphi$ has a local minimum at η, we have to prove that $\varphi\leq v$ in a neighborhood of η. Let us define the function

$$g(r):=\int_0^{2r}\varepsilon(s)ds.$$

We see that $g(0)=g'(0)=0$ and

$$g(r)\geq\int_r^{2r}\varepsilon(s)ds\geq r\varepsilon(r).$$

Then, by (51),

$$\begin{aligned}\varphi(\zeta)&=v(\eta)+\langle\zeta-\eta,\alpha\rangle-g(\|\zeta-\eta\|_{-1})\\&\leq v(\eta)+\langle\zeta-\eta,\alpha\rangle-\|\zeta-\eta\|_{-1}\cdot\varepsilon(\|\zeta-\eta\|_{-1})\leq v(\zeta),\quad\forall\zeta\in\mathcal{D}(V).\end{aligned}$$

Moreover, recalling that $(A^{-1})^\star=(A^\star)^{-1}$,

$$\nabla\varphi(\zeta)=\begin{cases}\alpha-(A^\star)^{-1}g'(\|\zeta-\eta\|_{-1})\dfrac{A^{-1}(\zeta-\eta)}{\|A^{-1}(\zeta-\eta)\|}, & \text{if } \zeta\neq\eta,\\ \alpha, & \text{if } \zeta=\eta.\end{cases}$$

This expression of $\nabla\varphi$ shows that $\zeta\mapsto A^\star\nabla\varphi(\zeta)$ is continuous. Therefore, φ is a test function. Finally, $\nabla\varphi(\eta)=\alpha$ and the proof is complete. □

Now we can state and prove our main result.

Theorem 4.6. *Let $v:\mathcal{D}(V)\to\mathbb{R}$ be concave, strictly increasing with respect to the component η_0 and $\|\cdot\|_{-1}$-continuous. Moreover, assume that v is a viscosity solution of (43) on $\mathcal{D}(V)$. Then v is differentiable along the direction $\hat{n}=(1,0)$ at any point $\eta\in\mathcal{D}(V)$ and the function $\eta\mapsto v_{\eta_0}(\eta)$ is continuous on $\mathcal{D}(V)$.*

Proof. Let $\eta \in \mathcal{D}(V)$ and $\alpha, \beta \in D^*v(\eta)$. Thanks to Proposition 3.12-(4), there exist sequences $(\eta_n), (\tilde{\eta}_n) \subset \mathcal{D}(V)$ such that:

- $\eta_n \to \eta$, $\tilde{\eta}_n \to \eta$;
- $\nabla v(\eta_n)$ and $\nabla v(\tilde{\eta}_n)$ exist for all $n \in \mathbb{N}$;
- $A^\star \nabla v(\eta_n) \rightharpoonup A^\star \alpha$ and $A^\star \nabla v(\tilde{\eta}_n) \rightharpoonup A^\star \beta$.

Since v is a viscosity solution of (43), thanks to Lemma 4.5 we can write, for any $n \in \mathbb{N}$,

$$\rho v(\eta_n) = \langle \eta_n, A^\star \nabla v(\eta_n)\rangle + f(\eta_n) v_{\eta_0}(\eta_n) + U_2(\eta_{0,n}) + \mathcal{H}(v_{\eta_0}(\eta_n)),$$

$$\rho v(\tilde{\eta}_n) = \langle \tilde{\eta}_n, A^\star \nabla v(\tilde{\eta}_n)\rangle + f(\tilde{\eta}_n) v_{\eta_0}(\tilde{\eta}_n) + U_2(\eta_{0,n}) + \mathcal{H}(v_{\eta_0}(\tilde{\eta}_n)).$$

Passing to the limit we get

$$\langle \eta, A^\star \alpha\rangle + f(\eta)\alpha_0 + U_2(\eta_0) + \mathcal{H}(\alpha_0) = \rho v(\eta) = \langle \eta, A^\star \beta\rangle + f(\eta)\beta_0 + U_2(\eta_0) + \mathcal{H}(\beta_0). \quad (52)$$

On the other hand, due to (36), we have $D^*v(\eta) \subset D^+v(\eta)$. Therefore $\alpha, \beta \in D^+(\eta)$. Since $D^+v(\eta)$ is convex, we have also $\lambda\alpha + (1-\lambda)\beta \in D^+v(\eta)$, for any $\lambda \in (0,1)$. So, by Lemma 4.5-(1), we have the subsolution inequality

$$\rho v(\eta) \le \langle \eta, A^\star(\lambda\alpha + (1-\lambda)\beta)\rangle + f(\eta)(\lambda\alpha_0 + (1-\lambda)\beta_0) + U_2(\eta_0) + \mathcal{H}(\lambda\alpha_0 + (1-\lambda)\beta_0), \ \ \forall \lambda \in (0,1). \quad (53)$$

Combining (52) and (53) we get

$$\mathcal{H}(\lambda\alpha_0 + (1-\lambda)\beta_0) \ge \lambda\mathcal{H}(\alpha_0) + (1-\lambda)\mathcal{H}(\beta_0).$$

Due to (38), we have $\alpha_0, \beta_0 \in D^+_{\hat{n}} v(\eta)$. Therefore, recalling that v is concave and strictly increasing with respect to the η_0 component, we have $\alpha_0, \beta_0 > 0$. Then the fact that $\mathcal{H}$ is strictly convex on $(0, +\infty)$ and the previous inequality show that $\alpha_0 = \beta_0$. This means that the projection of $D^*v(\eta)$ onto $\hat{n}$ is a singleton. Thanks to (36) this implies that also the projection of $D^+v(\eta)$ onto $\hat{n}$ is a singleton. Due to (38), we have that $D^+_{\hat{n}} v(\eta)$ is a singleton too. Since v is concave, this is enough to conclude that it is differentiable along the direction $\hat{n}$ at η.

We prove now that the map $\eta \mapsto v_{\eta_0}(\eta)$ is continuous on $\mathcal{D}(V)$. Let $\eta \in \mathcal{D}(V)$ and let $(\eta^n) \subset \mathcal{D}(V)$ be a sequence such that $\eta^n \to \eta$. We have to show that $v_{\eta_0}(\eta^n) \to v_{\eta_0}(\eta)$. Since v is concave, again by (38) for every $n \in \mathbb{N}$ there exists $p_1^n \in L^2_T$ such that $(v_{\eta_0}(\eta^n), p_1^n) \in D^+v(\eta^n)$. Being v concave, the set-valued map $\zeta \mapsto D^+v(\zeta)$ is locally bounded. Therefore, from any subsequence (η^{n_k}), we can extract a sub-subsequence $(\eta^{n_{k_h}})$ such that $\big(v_{\eta_0}(\eta^{n_{k_h}}), p_1^{n_{k_h}}\big)$ is weakly convergent towards some limit point. Due to the concavity of v, the set valued map $\eta \mapsto D^+v(\eta)$ is norm-to-weak upper semicontinuous (see [28], Chapter 1, Proposition 2.5). As consequence of this fact, this limit point must live in the set $D^+v(\eta)$. By (38) the limit point of $\big(v_{\eta_0}(\eta^{n_{k_h}})\big)$ must coincide with $v_{\eta_0}(\eta)$. This holds true for any subsequence $(v_{\eta_0}(\eta^{n_k}))$, so that the claim follows by the usual argument on subsequences. □

Remark 4.7. Note that in Theorem 4.6 we do not require that v is the value function: we require only that it is a concave, strictly increasing with respect the component η_0 and $\|\cdot\|_{-1}$-continuous viscosity solution of (43). In particular these properties are fulfilled by the value function (Propositons 2.14, 2.16, 3.8-(2) and Theorem 4.4). ■

Remark 4.8. The regularity result of the previous theorem allows us to define the feedback map

$$C(\eta) := \operatorname{argmax}_{c\geq 0}\left(U_1(c) - cV_{\eta_0}(\eta)\right), \quad \eta \in \mathcal{D}(V). \tag{54}$$

At least formally, this map defines an optimal feedback strategy for the problem. The study of it will be the subject of a forthcoming paper. ■

Remark 4.9. When the delay is concentrated at a point in a linear way, we might be tempted to insert the delay term in the infinitesimal generator A and try to proceed as in Section 3. Unfortunately this is not possible. Indeed consider a simple example:

$$\begin{cases} y'(t) = ry(t) + y\left(t-T\right), \\ y(0) = \eta_0, \ y(s) = \eta_1(s), \ s \in [-T, 0), \end{cases}$$

In this case we can define

$$A : \mathcal{D}(A) \subset H \longrightarrow H, \qquad (\eta_0, \eta_1(\cdot)) \longmapsto (r\eta_0 + \eta_1(-T), \eta_1'(\cdot)).$$

where again

$$\mathcal{D}(A) := \{\eta \in H \mid \eta_1(\cdot) \in W_T^{1,2}, \ \eta_1(0) = \eta_0\}.$$

The inverse of A is the operator

$$A^{-1} : (H, \|\cdot\|) \longrightarrow (\mathcal{D}(A), \|\cdot\|) \qquad (\eta_0, \eta_1(\cdot)) \longmapsto \left(\frac{\eta_0 - c}{r}, \ c + \int_{-T}^{\cdot} \eta_1(\xi)d\xi\right),$$

where

$$c = \frac{1}{r+1}\eta_0 - \frac{r}{r+1}\int_{-T}^{0} \eta_1(\xi)d\xi.$$

In this case we would have the first part of Lemma 3.7, but not the second part, because it is not possible to control $|\eta_0|$ by $\|\eta\|_{-1}$. Indeed, take for example r such that $\frac{1-r}{1+r} = \frac{1}{2}$, and $(\eta^n)_{n\in\mathbb{N}} \subset H$ such that

$$\eta_0^n = 1/2, \qquad \int_{-T}^{0} \eta_1^n(\xi)d\xi = 1, \quad n \in \mathbb{N}.$$

We would have $c = 1/2$, so that $\left|\frac{\eta_0^n - c}{r}\right| = 0$. Moreover we can choose η_1^n such that, when $n \to \infty$,

$$\int_{-T}^{0} \left|\frac{1}{2} + \int_{-T}^{s} \eta_1^n(\xi)d\xi\right|^2 ds \longrightarrow 0.$$

Therefore, we would have $|\eta_0^n| = 1/2$ and $\|\eta^n\|_{-1} \to 0$. This shows that the second part of Lemma 3.7 does not hold. Once this part does not hold, then everything in the following argument breaks down. ■

5 The optimal control problem with the state constraint $x(\cdot) \geq 0$

So far, we have considered the optimal control problem with state constraint $x(\cdot) > 0$. It is meaningful to consider the problem imposing a weaker state constraint $x(\cdot) \geq 0$. We will briefly explain which of the previous results remain true.

First of all the set of admissible strategies is wider and given in (5). The domain $\mathcal{D}(\bar{V})$ of the corresponding value function $\bar{V}$ is wider as well, as it contains also boundary points. Moreover, it may happen that the restriction of $\bar{V}$ to $\mathcal{D}(V)$ does not coincide with V. In general we only have the inequality $\bar{V} \geq V$ on $\mathcal{D}(V)$. However many properties of V can be proved for $\bar{V}$ as well with the same proofs. In particular, we would have:

- $\mathrm{Int}(\mathcal{D}(\bar{V}))$ is a convex $\|\cdot\|_{-1}$-open set of H and $\bar{V}$ is concave and $\|\cdot\|_{-1}$-continuous on this set;
- the monotonicity properties of Propositions 2.15, 2.16 are true for $\bar{V}$ as well;
- the regularity result of Theorem 4.6 holds for $\bar{V}$ on $\mathrm{Int}(\mathcal{D}(\bar{V}))$.

On the other hand, the continuity at the boundary $\partial\mathcal{D}(\bar{V})$ is not guaranteed and, in any case, not easy to prove. This is due to the fact that the boundary is not absorbing for the problem that is there exist points on the boundary and admissible strategies associated to such points leading the corresponding state into the interior region. This makes the study of the continuity at the boundary difficult.

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